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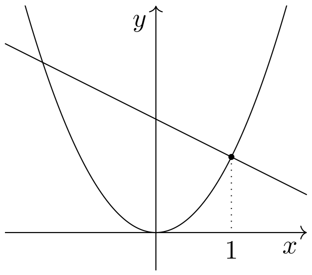
1401. Find the six roots of $\cos 3\theta = \frac{\sqrt{2}}{2}$ in $[0, 2\pi)$.

1402. The graph shows the curve $y = x^2$, and a normal to it at $x = 1$.



Show that the normal crosses the curve at $x = -\frac{3}{2}$.

1403. A quadrilateral has sides of length $1, 3, 3, 5$, not necessarily in that order. State, with a reason, whether it is possible for the quadrilateral to

(a) be an isosceles trapezium,

(b) have an interior right-angle,

(c) be cyclic.

1404. If $u = \tan 2\theta$, find $\dfrac{du}{d\theta}$.

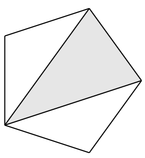
1405. A sample $\{x_i\}$ is taken, and the sample mean $\bar{x}$ is calculated. Afterwards, a quarter of the x_i values are increased by 10%, and the rest are reduced by 10%. Find the expected percentage change in $\bar{x}$.

1406. Two distinct quadratic functions f and g have $\mathrm{f}(a) = \mathrm{g}(a)$ and $\mathrm{f}'(a) = \mathrm{g}'(a)$ for some $a \in \mathbb{R}$. Prove that the equation $\mathrm{f}(x) = \mathrm{g}(x)$ has no other root but $x = a$.

1407. Solve the equation $\dfrac{4}{\sqrt{x}} = \sqrt{x}(x + 3)$.

1408. Prove that the sum of three consecutive squares is never a multiple of 3.

1409. The regular pentagon below has side length 2.



Show that the area of the shaded triangle is

$$A = \operatorname{cosec} \tfrac{\pi}{5} + \cot \tfrac{\pi}{5}.$$

1410. A quadratic function is $\mathrm{f}(x) = ax^2 + bx + c$, where $a, b, c \in \mathbb{R}$ and $a \neq 0$. Show that, for all x,

$$\mathrm{f}\left(-\tfrac{b}{2a} - x\right) = \mathrm{f}\left(-\tfrac{b}{2a} + x\right).$$

1411. Show that the iteration $a_{n+1} = \sqrt{12a_n - 36}$ has exactly one fixed point, and determine its value.

1412. Disprove the following statement:

$$\text{If } \mathrm{f}(x) \not\equiv \mathrm{g}(x), \text{ then } \mathrm{f}^2(x) \not\equiv \mathrm{g}^2(x).$$

1413. A student comes across an apparent "proof" that $1 = 0$. It runs as follows:

Let $x = 1$.
This gives $x^2 - x = 0$.
So $x(x - 1) = 0$.
Dividing by $(x - 1)$ yields $x = 0$.
Hence $1 = 0$.

Explain precisely where the logic breaks down.

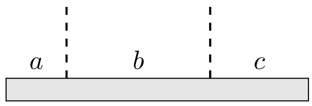
1414. Two dice are rolled. Find the probability that the difference between the scores is odd.

1415. A function f is defined over the reals, and has range $[-a, a]$. Give the range of each of the following:

(a) $x \mapsto \frac{1}{a}\mathrm{f}(x)$,

(b) $x \mapsto \frac{1}{a^2}\big(\mathrm{f}(x)\big)^2$.

1416. A strip light, which is modelled as a uniform rod, is suspended from two vertical chains.



These have been set up asymmetrically, dividing the length of the strip light in the ratio $a : b : c$, where $a < c$.

(a) Explain how you know that $\dfrac{c}{a + b + c} \leq \frac{1}{2}$.

(b) Show that the ratio of tensions is

$$a + b - c : c + b - a.$$

1417. Evaluate $\log_p q^2 \times \log_q p^3$.

1418. Find the exact angle between the hands of a clock at 3:20pm. Give your answer in radians.

1419. Show that no (x, y) points satisfy both of

$$x^2 + y^2 = 1,$$
$$2x + 3y > 4.$$

1420. The surd expression $x^{\frac{1}{3}} + 1$, where $x \in \mathbb{Z}$, may be rationalised by multiplication by a factor of the form $x^{\frac{2}{3}} + px^{\frac{1}{3}} + q$. Find p and q.

1421. Solve the equation $\Big[ax^2 - a^2x\Big]_{x=1}^{x=2} = 0$.

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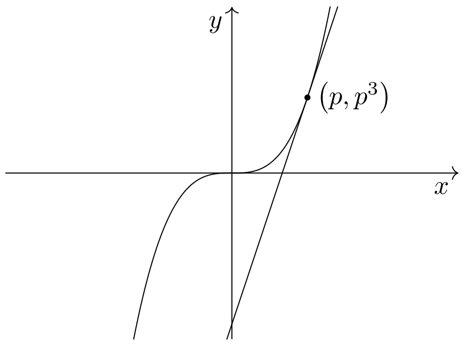
1422. State, giving a reason, which of the implications $\Longrightarrow$, $\Longleftarrow$, $\Longleftrightarrow$ links the following statements concerning a polynomial function f:

① $(x-\alpha)$ does not divide exactly into $\mathrm{f}(x)$,
② $\mathrm{f}(\alpha) \neq 0$.

1423. The interior angles of a quadrilateral form an AP. Give, in radians, the set of possible values for the second largest angle.

1424. On the curve $y = x^3$, a generic tangent line is drawn at $x = p$:



The tangent line has equation $y = mx + c$.

(a) Find m in terms of p.
(b) Hence, find c in terms of $x = p$.

1425. A function h has domain $\mathbb{R}$ and range $[0, 1]$. State, with a reason, whether the following hold for all constants c:

(a) $\mathrm{h}(x) + c \in [c, c+1]$ for all $x \in \mathbb{R}$,
(b) $x \mapsto \mathrm{h}(x) + c$ has range $[c, c+1]$ over $\mathbb{R}$.

1426. An error has been found in a statistical procedure: a zero was missed off the end of the quantity $\sum x^2$. State whether the following will increase, decrease or neither when the error is corrected:

(a) the mean,
(b) S_{xx},
(c) the standard deviation.

1427. An inequality is given as

$$(x^2 + 5)(x - 1)(3x + 2) \geq 0.$$

By considering the signs of the three factors, or otherwise, solve the inequality.

1428. Simplify the following:

(a) $A \setminus \big(A \cap B'\big)$.
(b) $\big(A \cap B'\big) \cup \big(A' \cap B\big) \cup \big(A' \cap B'\big)$.

1429. Show that, if $\log_x y + \log_y x = 2$, then $x = y$.

1430. Give a counterexample to the following claim: "If a population is bimodal, then its median and mode cannot be equal."

1431. Shade the region of the (x, y) plane which satisfies both of the following inequalities:

$$y < 2x, \qquad 4x^2 + 4y^2 < 1.$$

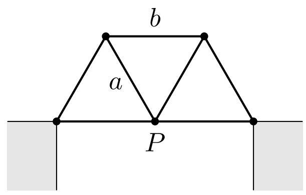
1432. Show that the outputs of $\mathrm{h}(x) = 4 - (x-5)^2$ are always exceeded by its inputs.

1433. You are given that the graphs $y = x^2 + 2x + 3$ and $y = x^2 + px + q$ intersect. Find all possible values for the constants p, q. Give your answer as two statements in set notation.

1434. In a gliding competition, the challenge is to travel around the triangle of greatest area. The winner performs three legs, of distances 10, 17 and 21 km. Find the area of the triangle.

1435. Show that the half-parabola $y = \sqrt{x}$ and the line $2y\sqrt{a} = x + a$ are tangent at $x = a$.

1436. A bridge over a stream is built of seven timber beams, which are modelled as rods of negligible thickness. The beams may be assumed to be light compared to a heavy load carried by the bridge, which is modelled as being located at point P.



Explain how you know, without any calculations, that the beam marked

(a) a must be in tension,
(b) b must be in compression.

1437. The mappings below show the domains and ranges of two functions f and g:

$$\mathrm{f} : A \longmapsto B,$$
$$\mathrm{g} : C \longmapsto D.$$

State, with a reason, what must be true of the sets A, B, C, D if fg is to be a well-defined function.

1438. The curved surface of a cone of radius r and height h is unwrapped and laid flat, forming a sector. Prove that the angle ϕ subtended by this sector, given in radians, is

$$\phi = \frac{2\pi r}{\sqrt{r^2 + h^2}}.$$

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1439. For constants $a < b < c < d$, prove that

$$\int_a^c y\,dx + \int_b^d y\,dx = \int_a^d y\,dx + \int_b^c y\,dx.$$

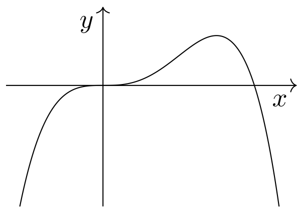
1440. In the positive quadrant, a half-parabola is defined as $y = x^2 + 1$ for $x \geq 0$. This section forms part of a pattern with rotational symmetry order four, centred on the origin. The section in the quadrant $x \geq 0, y \leq 0$ has equation $y = -\sqrt{x - 1}$.

(a) Sketch the pattern.

(b) Give the equation of each of the other sections, in the form $y = \mathrm{f}(x)$.

1441. Solve $\dfrac{1 - \frac{1}{2x}}{\sqrt{1 - \frac{1}{4x^2}}} = 2.$

1442. The diagram shows the graph $y = \mathrm{f}(x)$, where f is a quartic function.



State, with a reason, whether each of the following could be the definition of the function f:

(a) $\mathrm{f}(x) = 2x - x^4$,

(b) $\mathrm{f}(x) = 2x^2 - x^4$,

(c) $\mathrm{f}(x) = 2x^3 - x^4$.

1443. Show that $\displaystyle\int_0^1 15\left(x + \sqrt{x}\right)\left(1 - \sqrt{x}\right)\,dx = 4.$

1444. Either prove or disprove the following: "If forces with magnitudes $\frac{3}{5}Q$, $\frac{4}{5}Q$, Q act on an object, and it remains in equilibrium, then two of the forces must be perpendicular."

1445. Express $x^2 + 6x - 8$ as the product of two linear factors containing surds.

1446. In a casino, a short-sighted croupier is calling rolls of a die. Each roll, there is a 75% probability that he reads the score correctly, and a 25% probability that he misreads the score, changing it by one. If both options are available, then he is equally likely to over-read or under-read the score. He always calls one of the numbers $\{1, 2, 3, 4, 5, 6\}$.

Show that the probability of the croupier calling 6 has been reduced to $\frac{7}{48}$ by his short-sightedness.

1447. Three statements regarding an integer n are

$$\begin{aligned} ①&: n \in (\infty, 0] \\ ②&: 2 < n < 5 \\ ③&: n^2 > 9 \end{aligned}$$

Find the set of integers which satisfy **none** of the above statements.

1448. The function $\mathrm{S}(n)$ is defined, for a sequence u_i, by

$$\mathrm{S}(n) = \sum_{i=1}^{n} u_i.$$

Find the following sums, giving your answers in terms of the function S and the variable n:

(a) $\displaystyle\sum_{i=1}^{n}(u_i + 1)$,

(b) $\displaystyle\sum_{i=1}^{n}(2u_i + i)$.

1449. Two particles are in motion, moving in 1D. At time t, their positions on this axis are given by

$$x_1 = \frac{1}{1 - t}, \qquad x_2 = \frac{2}{1 - 2t}.$$

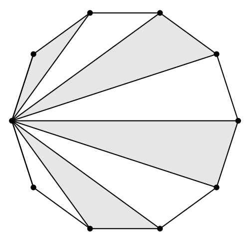
Determine whether they collide.

1450. Assume, for a contradiction, that $\log_2 3$ is rational, and can therefore be written as $\frac{a}{b}$, where $a, b \in \mathbb{N}$.

(a) Show that $2^a = 3^b$.

(b) Hence, by considering prime factors, prove that $\log_2 3$ is irrational.

1451. In the diagram below, four shaded triangles have been constructed inside a regular decagon.



Prove that half of the decagon is shaded.

1452. Variable z is defined as $z = x^5 - 2x^4 + x^3$. Find all values of x for which z is stationary with respect to x.

1453. A function g has $\mathrm{g}'(x) = a$ for all $x \in \mathbb{R}$, where a is a constant. Prove that, for any sample of real numbers $\{x_i\}$, the mean of $\{\mathrm{g}(x_i)\}$ is $\mathrm{g}(\bar{x})$.

1454. By setting up and solving simultaneous equations, express the fraction $\frac{36}{323}$ in the form ${}^1/_a + {}^1/_b$, where a and b are distinct prime numbers.

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1455. The masses of a population of rodents are modelled with a normal distribution, in units of grams:

$$M \sim N\left(245, 120^2\right).$$

Two such rodents are caught in the same trap.

(a) Assuming the model, find the probability that

i. both weigh more than 245 grams,

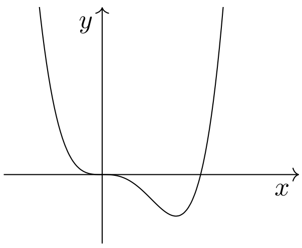
ii. neither weighs more than 250 grams.

(b) Give two reasons why probabilities calculated in this fashion are unlikely to be useful to a biologist.

1456. Solve $\sin\left(2x + \frac{\pi}{3}\right) = \cos\left(2x + \frac{\pi}{3}\right)$ for $x \in [0, 2\pi]$.

1457. A student writes: "When you step on the pedals of a bicycle, a frictional force is generated, which acts backwards on the back wheel of the bicycle." Explain whether this is correct.

1458. The curve $y = x^4 - x^3$ is shown below:



(a) Find the second derivative.

(b) Determine whether the second derivative has a sign change at the origin.

(c) A neighbourhood of the origin is defined by the interval $[-1/4, 1/4]$. Determine whether the curve is concave, convex or neither over this interval.

1459. Let a, b, c and d be distinct positive real constants. Write down the roots of the following equation:

$$\frac{(x^3 - a^3)(x^3 + b^3)}{(x^3 - c^3)(x^3 + d^3)} = 0.$$

1460. You are given that the sequence $\{a_n\}$ is arithmetic. Sequence $\{b_n\}$ is defined by $b_n = 10^{a_n}$. Prove that $\{b_n\}$ is a geometric sequence.

1461. Three playing cards are dealt, with replacement, from a standard deck. Determine which, if either, of the following events has the greater probability: the three cards being, in any order,

- an ace, a king and a queen,
- two fours and a five.

1462. Two vectors $\mathbf{p}$ and $\mathbf{q}$ are given in terms of the usual perpendicular unit vectors as

$$\begin{aligned}\mathbf{p} &= 3\mathbf{i} + 4\mathbf{j},\\ \mathbf{q} &= 4\mathbf{i} - 3\mathbf{j}.\end{aligned}$$

(a) Give the magnitudes of $\mathbf{p}$ and $\mathbf{q}$,

(b) Express $\mathbf{i}$ and $\mathbf{j}$ in terms of $\mathbf{p}$ and $\mathbf{q}$.

1463. Show that the following equation defines a family of circles:

$$\int x\,dx + \int y\,dy = 0.$$

1464. The scores a and b on two exams, out of A and B marks respectively, are to be combined into one mark M, given out of a hundred. The exams (as opposed to the individual marks) are to have equal weighting. Find a formula for M.

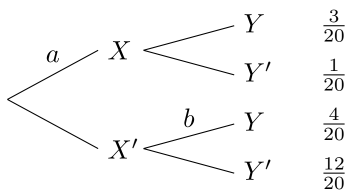
1465. Sketch $y = |\tan x|$, using units of degrees.

1466. Show that, if the following simultaneous equations are to be satisfied, then $z = 2$.

$$\begin{aligned}x - y + z &= 3,\\ 2x - z &= 2y.\end{aligned}$$

1467. Show that the discriminant of $y = (px+1)(qx+1)$ is $(p-q)^2$.

1468. Events X and Y have probabilities as represented on the following tree diagram:



(a) Find probabilities a and b.

(b) Find $\mathbb{P}(X \cup Y')$.

1469. In this question, do not use a polynomial solver.

By first locating a root with the Newton-Raphson method, determine the solution of

$$0.04x^3 - 0.07x^2 = 0.62x + 0.15.$$

1470. Write down $\displaystyle\int e^x + e^{-x}\,dx$.

1471. Two curves are given as

$$y = 9 - 2\left(x-3\right)^2, \quad y = 2x^2.$$

By calculating the discriminant of an equation, show that these curves are tangent.

1472. Four values x_1, x_2, x_3, x_4 are chosen at random and independently from the interval $[0, 1]$. Write down the probability that $x_1 + x_2 < x_3 + x_4$.

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1473. State, with a reason, whether the argument below, concerning a trio of statements $\{A, B, C\}$, holds: "If $A \implies B$ and $C \implies B$, then $A \implies C$."

1474. A mass is in equilibrium with three forces acting on it. These forces have magnitudes $5, 10$ and 12 N. Determine the obtuse angle between the 10 and 12 N forces.

1475. Solve the following simultaneous equations:

$$\begin{aligned} x + y + z &= 0, \\ x + y - z &= 1, \\ x - y + z &= 2. \end{aligned}$$

1476. Prove algebraically that, for any set of data,

$$\sum_{i=1}^{n} (x_i - \bar{x}) = 0.$$

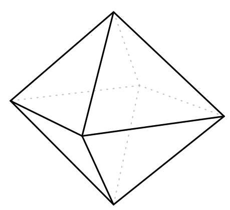
1477. The quadratic functions f, g have discriminants which are zero and positive respectively. For each of the following equations, write down the possible numbers of distinct real roots:

(a) $\mathrm{f}(x)\,\mathrm{g}(x) = 0$,

(b) $\mathrm{f}(x)\big(\mathrm{g}(x)\big)^2 = 0$,

(c) $\big(\mathrm{f}(x)\big)^4\big(\mathrm{g}(x)\big)^7 = 0$.

1478. A regular octahedron is shown below.



Two of the faces of the octahedron are selected at random. Find the probability that these faces share an edge.

1479. Simplify $(e^x + 1)^4 - (e^x - 1)^4$.

1480. General linear simultaneous equations are given below, for constants a, b, c, d, p, q with $ad \neq bc$.

$$\begin{aligned} ax + by &= p, \\ cx + dy &= q. \end{aligned}$$

(a) Solve to find formulae for x and y.

(b) Explain the need for the condition $ad \neq bc$.

1481. Show that $\dfrac{1}{1 + \sqrt[4]{2}} = 2^{\frac{1}{4}} - 2^{\frac{1}{2}} + 2^{\frac{3}{4}} - 1$.

1482. A parabola is given by $y = ax^2 + bx + c$. Write down the equations of the graphs obtained when the original parabola is

(a) reflected in the x axis,

(b) reflected in the y axis,

(c) reflected in the line $y = x$.

1483. By factorising, solve the equation

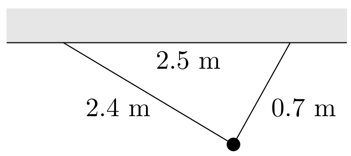
$$(x + 2)^3(x - 1) + (x + 2)(x - 1)^3 = 0.$$

1484. Two dice are rolled; neither shows a six. Find the probability that the combined score is at least six.

1485. If $a = 2^{x+1}$ and $b = 4^{x-1}$, write a in terms of b.

1486. Show algebraically that $18x^3 + 33x^2 = 7x + 2$ has three roots, all of which are rational.

1487. A 10 kg audio rig, modelled as a particle, is hung from two cables, lengths 0.7 m and 2.4 m, which are attached to two points, 2.5 m apart, on the horizontal ceiling of a concert space.



(a) Find the angles of inclination of the ropes.

(b) Using a triangle of forces, or otherwise, find the tensions in the two ropes.

1488. Show that $\cos(\arcsin x) \equiv \sqrt{1 - x^2}$.

1489. If $\frac{d}{dx}\left(x^2 + y^2\right) = 0$, find $\frac{dy}{dx}$ in terms of x and y.

1490. Find the value of the constant b, if the following may be written as a quadratic function of x:

$$\frac{2x^3 - x^2 + bx - 3}{2x + 1}.$$

1491. A set of forty data is summarised with mean $\bar{x}$ and standard deviation $s \neq 0$. A check shows, however, that the lowest five data points were, in fact, duplicated, and only thirty-five data should have been recorded. State, with a reason, what effect rectifying this mistake will have on

(a) the sample mean,

(b) the sample standard deviation.

1492. A locus is defined, in terms of two points $A : (1, 3)$ and $B : (5, 1)$, as the set of points P for which $|AP| = |BP|$. Determine the Cartesian equation of the locus.

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1493. Show that the equation

$$\frac{1}{x} + \frac{1}{x+1} + \frac{1}{x+2} = 0$$

has precisely two distinct real roots.

1494. Evaluate $\displaystyle\sum_{i=1}^{50} \log_8 \left(2^i\right)$.

1495. Find constants $a, b, c, d > 0$ such that

$$(a + bx)^3 \equiv c + 225x + dx^2 + 27x^3.$$

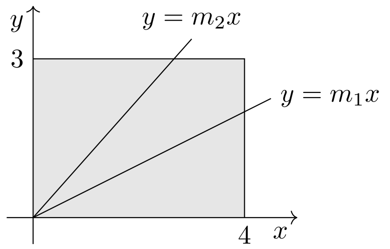
1496. Describe the transformation that takes the graph $y = (x-p)^2 + q$ onto the graph $y = (x-p+1)^2 + q$.

1497. Find formulae for the sum of

(a) the first n even integers,

(b) the first n odd integers.

1498. A rectangle has vertices at $(0, 0)$, $(4, 0)$, $(0, 3)$ and $(4, 3)$. Two lines $y = m_1 x$ and $y = m_2 x$ are drawn, which divide the rectangle into three regions of equal area.



(a) Show that $m_1 = \frac{1}{2}$.

(b) Determine m_2.

1499. Find the probability that six consecutive dice rolls give six different scores.

1500. Prove that regular heptagons do not tessellate.

——— End of 15th Hundred ———

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